INFLUENCE OF RUBBER BUSHINGS ON STAY-CABLE DAMPER EFFECTIVENESS

J.A. Main and N.P. Jones
University of Illinois at Urbana-Champaign, Department of Civil and Environmental Engineering
205 N. Mathews Ave, Urbana, IL 61801
jmain@uiuc.edu, npjones@uiuc.edu

Abstract

To suppress stay-cable vibrations, external dampers are commonly attached to the cables transversely near the anchorages. To reduce bending stresses, neoprene rubber bushings are commonly also mounted on the cables inside steel guide pipes at the anchorages. The influence of these bushings on damper effectiveness is investigated by modeling the stay-cable as a taut string, the bushing as a linear spring, and the damper as a linear dashpot, and an exact analytical formulation for free vibrations of this system is developed. An iterative scheme for numerical computation of the complex eigenfrequencies is presented, and an asymptotic approximate solution is obtained which enables a generalization of the design curve for a linear viscous damper to account for the bushing stiffness.

INTRODUCTION

Stay-cable vibrations have been observed on numerous cable-stayed bridges around the world, and mitigation of these vibrations has become a significant concern in the design and retrofit of cable-stayed bridges. To reduce bending stresses at the anchorages, neoprene rubber bushings are commonly installed on stay cables inside steel guide pipes at the anchorages. Such bushings provide additional transverse stiffness near the anchorage, but their energy dissipation capability is usually negligible. Stay cables have very low levels of inherent damping, and in order to reduce their susceptibility to vibration, it is desirable to provide additional energy dissipation. For this purpose, supplemental damping devices, such as fluid dampers, are commonly attached to the stay cables. Such dampers are attached transversely to the cables, somewhat further along the span than the rubber bushings, but still quite close to the anchorages.

Because stay cables typically have very large axial tension, their dynamics can be well approximated in many cases using a taut-string model. Modeling the cable-damper system as a taut string with a linear viscous damper attached near the end, a universal design curve has been developed to estimate the amount of supplemental damping provided in the first few modes of vibration as a function of the external damper coefficient ([1]). This curve was originally developed using an approximate numerical formulation, but recently, an exact analytical formulation has been developed ([2], [3]), yielding an explicit equation for the universal curve from asymptotic approximations. While very useful for design purposes, this universal curve does not account for the presence of the rubber bushing between the damper and the anchorage. Takano et al. [4] found that the stiffness of the bushing could significantly reduce the damper effectiveness, requiring the damper to be attached further along the cable span to achieve the same performance.

In this paper, vibrations of a stay cable with rubber bushing and external damper are investigated by modeling the stay cable as a taut string, the damper as a linear dashpot, and the bushing as a linear spring attached between the dashpot and the end of the cable. An exact analytical solution of the free vibration problem is formulated, an iterative equation is developed to facilitate numerical evaluation of the complex eigenfrequencies, and an explicit asymptotic solution is derived, which is quite accurate in the first few modes for damper locations near the end of the cable. It is observed that the
influence of the stiffener can be interpreted as reducing the effective distance of the damper from the end of the cable. An explicit asymptotic expression for this effective damper location is derived as a function of the bushing location and the nondimensional bushing stiffness, and using this effective damper location, the universal curve of [1] is generalized to account for the influence of the bushing.

PROBLEM FORMULATION

The problem under consideration is depicted in Fig. 1. A linear dashpot with viscous coefficient $c$ and a linear spring with stiffness $k$ are attached to a taut cable at intermediate points, dividing the cable into three segments. In addition to the usual spanwise coordinate $x$, the coordinates $x' = L - x$ and $x'' = x - x_k$ will be used for convenience in expressing the solution and are indicated in Fig. 1.

![Fig. 1: Taut cable with attached spring and dashpot.](image)

Assuming that the tension in the cable is large compared to its weight, bending stiffness and internal damping are negligible, and deflections are small, the following partial differential equation is satisfied over each segment of cable:

$$m\ddot{y} = Ty''$$  \hspace{1cm} (1)

The prime denotes differentiation with respect to $x$, and the dot denotes differentiation with respect to time $t$. This equation is violated at the attachment points of the spring and dashpot, at which point the applied forces must be balanced by a discontinuity in slope of the cable. Equilibrium of forces at the attachment points of the spring and dashpot can be written as follows:

$$T[y'(x_k^+) - y'(x_k^-)] = ky(x_k)$$  \hspace{1cm} (2)

$$T[y'(x_c^+) - y'(x_c^-)] = cy(x_c)$$  \hspace{1cm} (3)

The boundary conditions of zero displacement at the ends of the cable must also be satisfied, and the displacement of the cable must be continuous at the spring and dashpot locations:

$$y(0,t) = 0; \quad y(L,t) = 0$$  \hspace{1cm} (4)

$$y(x_k^+) = y(x_k^-); \quad y(x_c^+) = y(x_c^-)$$  \hspace{1cm} (5)

To solve (1) subject to (2)-(5), a separable solution in space and time is assumed of the form
\[ y(x, \tau) = Y(x)e^{i\omega \tau} \]  

(6)

in which a nondimensional time \( \tau = (\pi/L)\sqrt{T/m} \cdot t \) has been introduced using the fundamental frequency of a taut cable, so that the nondimensional eigenfrequency \( \omega \) takes on integer values in the absence of the spring and dashpot. Due to the influence of the dashpot, the nondimensional frequency \( \omega \) is complex in general, with its imaginary part giving the rate of decay of oscillation in time. Substituting the assumed form of solution into (1) yields the following ordinary differential equation:

\[ y'' + \beta^2 y = 0 \]  

(7)

where \( \beta = \pi\omega / L \) is the complex wave number. The solution to (7), subject to the boundary and continuity conditions in (4) and (5), can be written as

\[
Y(x) = \begin{cases} 
\alpha_k \frac{\sin(\beta x)}{\sin(\beta x_k)}, & 0 \leq x \leq x_k \\
\alpha_k \left[ \cos(\beta x^*) - \frac{\sin(\beta x^*)}{\tan(\beta x^*)} \right] + \alpha_c \frac{\sin(\beta x^*)}{\sin(\beta x_c^*)}, & x_k \leq x \leq x_c \\
\alpha_c \frac{\sin(\beta x^*)}{\sin(\beta x_c^*)}, & x_c \leq x \leq L
\end{cases}
\]  

(8)

where \( \alpha_k \) and \( \alpha_c \) denote the amplitude at the spring and damper, respectively, which are complex in general to allow for a difference in phase between the motions of the two points. Substituting this solution into the equilibrium equations (2) and (3) then yields the following set of equations:

\[
\begin{bmatrix}
\cot(\beta x_k) + \cot(\beta x^*) + \frac{k}{\beta T} & -\csc(\beta x^*) \\
-\sin(\beta x^*) - \frac{\cos^2(\beta x^*)}{\sin(\beta x_c^*)} & \cot(\beta x^*) + \cot(\beta x_c^*) + i\frac{c}{\sqrt{Tm}}
\end{bmatrix}
\begin{bmatrix}
\alpha_k \\
\alpha_c
\end{bmatrix} = 0
\]  

(9)

For nontrivial solutions the determinant of the matrix in (9) must vanish; setting this determinant equal to zero yields the following equation, after some simplification:

\[ \cot(\beta x_c) + \cot(\beta x_c^*) + K = -i\frac{c}{\sqrt{Tm}} \]  

(10)

where \( K \) represents the influence of the spring in the solution and is defined as follows:

\[
K = \frac{(k / \beta T)[1 + \cot^2(\beta x_k)]}{[1 + \cot^2(\beta x_k)] + (k / \beta T)[\cot(\beta x_k) - \cot(\beta x_c)]}
\]  

(11)

When \( k = 0 \) (no spring), \( K = 0 \), and (10) reduces to the solution previously obtained in [2] and [3]. It is noted that the right-hand side of (10) is purely imaginary, whereas the terms on the left-hand side are complex in general. Taking the real part of (10) then yields an equation for the complex eigenfrequencies that is independent of the external damper coefficient \( c \), and the solution branches to this equation reveal the entire range of solution characteristics for given values of \( x_c / L, x_k / L \), and \( k \).
ITERATIVE SOLUTION

Apart from the term $K$, which represents the influence of the spring, (10) is identical to the equation previously derived for eigenfrequencies of a taut string with dashpot only. In [2] this equation was rearranged into a form suitable for iterative solution; grouping $K$ with the nondimensional damper coefficient and following a similar procedure, the following equation can be obtained:

$$
\tan(\beta L) = \frac{(i\eta + K)\sin^2(\beta x)}{1 + (i\eta + K)\sin(\beta x)\cos(\beta x)}
$$

(12)

where $\eta$ denotes the nondimensional damper coefficient:

$$
\eta = \frac{c}{\sqrt{Tm}}
$$

(13)

The $n^{th}$ nondimensional eigenfrequency $\omega_n = \beta_n L/\pi$ can then be computed iteratively by substituting the current estimate $\omega_n^j$ into the right-hand side of (12) and inverting the tangent function to obtain an improved estimate $\omega_n^{j+1}$, as follows:

$$
\omega_n^{j+1} = n + \frac{1}{\pi} \tan^{-1}\left[ \frac{(i\eta + K_n^j)\sin^2(\pi\omega_n^j x / L)}{1 + (i\eta + K_n^j)\sin(\pi\omega_n^j x / L)\cos(\pi\omega_n^j x / L)} \right]
$$

(14)

where $K_n^j$ from (11) can be expressed in terms of the current estimate $\omega_n^j$ as follows:

$$
K_n^j = \frac{[\chi/(\pi\omega_n^j x / L)][1 + \cot^2(\pi\omega_n^j x / L)]}{[1 + \cot^2(\pi\omega_n^j x / L)] + [\chi/(\pi\omega_n^j x / L)][\cot(\pi\omega_n^j x / L) - \cot(\pi\omega_n^j x / L)]}
$$

(15)

The nondimensional spring stiffness $\chi$ introduced in (15) is defined as

$$
\chi = \frac{kx}{T}
$$

(16)

The integer eigenfrequencies $\omega_n = n$ corresponding to $\chi = 0$ and $\eta = 0$ can be used as the initial guess $\omega_n^0$ in the iteration of (14), and the iterative scheme converges fairly quickly. The complex eigenfrequencies depend on four nondimensional parameters: $x / L$, $x / L$, $\eta$, and $\chi$. The nature of this dependence will be clarified by the asymptotic solution in the following section.

ASYMPTOTIC SOLUTION

Eq. (10) can be rearranged into the equivalent following form, which facilitates asymptotic solution:

$$
\sin(\beta L) + (k / \beta T)\sin(\beta x)\sin(\beta x') + i\eta\sin(\beta x)\sin(\beta x')
$$

$$
+ i(k / \beta T)\eta\sin(\beta x)\sin(\beta x')\sin(\beta x') = 0
$$

(17)
The complex eigenfrequencies can be written as follows

\[ \omega_n = n + \Delta \omega_n \]  

(18)

where \( \Delta \omega_n \) is a complex-valued frequency increment induced by the spring and dashpot. When \( n(x_c / L) \ll 1 \), then \( \Delta \omega_n \ll1 \), and the following approximations for the sinusoidal terms in (17) can be introduced:

\[
\begin{align*}
\sin(\beta L) & \equiv \pi(-1)^n \Delta \omega_n \\
\sin(\beta x_i) & \equiv \pi n(x_i / L), \quad \sin(\beta x_i') \equiv \pi(1)^n[\Delta \omega_n - n(x_i / L)] \\
\sin(\beta x_c) & \equiv \pi n(x_c / L); \quad \sin(\beta x_c') \equiv \pi(1)^n[\Delta \omega_n - n(x_c / L)] \\
\sin(\beta x_c'') & \equiv \pi n(x_c'' / L)
\end{align*}
\]

Taking these approximations into (17), neglecting higher-order terms, and solving for \( \Delta \omega_n \) then yields the following asymptotic approximation for the complex frequency increment:

\[
\Delta \omega_n \approx \frac{n}{L} \left[ \frac{\chi(x_i / x_c) + i \pi \eta n(x_c / L)[1 + \chi(1 - x_c / x_i)]}{1 + \chi} + i \pi \eta n(x_c / L)[1 + \chi(1 - x_c / x_i)] \right]
\]

(23)

in which the effective damper location \( \chi_c^{\text{eff}} \) has been introduced, which is defined as

\[ \chi_c^{\text{eff}} = \frac{x_c [1 + \chi(1 - x_c / x_i)]}{(1 + \chi)} \]

(26)

The first term in (24) represents the real-valued frequency increment induced by the spring alone, which will be denoted \( \Delta \omega_n^{(p=0)} \):

\[
\Delta \omega_n^{(p=0)} \equiv n \left( \frac{x_i}{L} \right) \frac{\chi}{1 + \chi}
\]

(27)

In Fig. 2(a), this asymptotic approximation for the real-valued frequency increment \( \Delta \omega_n^{(p=0)} \) induced by the spring is plotted as a function of the nondimensional spring stiffness \( \chi \) along with the exact values computed from (14), for \( n = 1, \ x_i / L = 0.015, \) and \( x_c / L = 0.02. \) As \( \eta \rightarrow \infty, \) the dashpot locks the cable at its attachment point, and it can be seen from (23) that \( \Delta \omega_n \) tends to the following purely real value, independent of the nondimensional spring stiffness \( \chi :\)
\[ \Delta \omega_n^{(q \to \infty)} \equiv n \left( \frac{x_c}{L} \right) \]  

(28)

(The exact value is \( \Delta \omega_n^{(q \to \infty)} = n(x_c / x'_c) \), which is asymptotically equivalent to (28) for \( x_c / L \ll 1 \).

The maximum contribution of the damper to the real part of the frequency increment is then given by

\[ \Delta \omega_n^{(q \to \infty)} - \Delta \omega_n^{(q=0)} \equiv n \left( \frac{x_c^{(\text{eff})}}{L} \right) \]  

(29)

Combining (24) and (25) and eliminating the parameter grouping \( (\pi n x_c^{(\text{eff})} / L) \) yields the following equation, which indicates the the complex frequency increment \( \Delta \omega_n \) traces a semi-circle in the complex plane with radius \( n(x_c^{(\text{eff})} / L) / 2 \):

\[
\{\text{Re}[\Delta \omega_n] - [\Delta \omega_n^{(q=0)} + n(x_c^{(\text{eff})} / L) / 2]\}^2 + \text{Im}[\Delta \omega_n]^2 \equiv [n(x_c^{(\text{eff})} / L) / 2]^2
\]  

(30)

Fig. 2(b) shows plots of the frequency increment \( \Delta \omega_n \) in the complex plane for different values of the nondimensional spring stiffness \( \chi \); curves corresponding to the asymptotic approximation (30) are plotted along with exact solution generated from (14). As \( \chi \) increases, the effective damper location \( x_c^{(\text{eff})} \) decreases according to (26), reducing the radius of the semi-circle formed by \( \Delta \omega_n \) in the complex plane according to (30), and leading to smaller values of \( \text{Im}[\Delta \omega_n] \), and consequently, smaller damping.

Fig. 2: Dependence of nondimensional complex frequency increment \( \Delta \omega_n \) on nondimensional spring stiffness \( \chi \) and nondimensional damper coefficient \( \eta \) \((n=1, \ x_c / L = 0.015, \ x_c / L = 0.02)\)
The special case of a dashpot and spring attached in parallel at the same position is called a Voigt element; vibrations of a taut cable with this type of damper were investigated numerically in [5]. Denoting the attachment point of the Voigt element as \( x_v \), the eigenvalue equation (17) for this case reduces to

\[
\sin(\beta L) + \left( \frac{k}{\beta T} + i \frac{c}{\sqrt{I_m}} \right) \sin(\beta x_v) \sin(\beta x_e') = 0
\]  

For this case, the asymptotic approximations (24) and (25) are still valid, with \( x_k = x_c = x_v \), and the effective dashpot location \( x_e^{(\text{eff})} \) in this case is given by

\[
x_e^{(\text{eff})} = \frac{x_v}{(1 + \chi)}
\]  

**GENERALIZATION OF UNIVERSAL CURVE**

When the complex frequency increment is small, \( \Delta \omega_n \ll 1 \), the damping ratio in a given mode \( \xi_n = \text{Im}[\omega_n]/|\omega_n| \) can be approximated as \( \xi_n \equiv \text{Im}[\omega_n]/n \). Using this approximation with the asymptotic solution for \( \text{Im}[\omega_n] \) in (25) yields the following asymptotic approximation for the modal damping ratios:

\[
\frac{\xi_n}{(x_e^{(\text{eff})}/L)} \equiv \frac{\pi \eta n x_e^{(\text{eff})}/L}{1 + (\pi \eta n x_e^{(\text{eff})}/L)^2}
\]  

This asymptotic relation is equivalent to that derived for the taut cable with dashpot alone in [2] except that the dashpot location \( x_c \) is replaced by the effective location dashpot location \( x_e^{(\text{eff})} \) given by (26). This suggests the following generalization of the universal design curve to account for the influence of a rubber bushing: for a bushing with location \( x_b/L \) and nondimensional stiffness \( \chi \), the effective viscous damper location can be computed from the asymptotic relation (26), as depicted in Fig. 3(a), and the modal damping ratios can then be estimated as a function of the nondimensional viscous damper coefficient \( \eta \) from the asymptotic relation (33), as depicted in Fig. 3(b).

![Fig. 3: Generalization of universal curve for taut cable with spring and dashpot: (a) Asymptotic relation (26) for effective dashpot location. (b) Asymptotic relation (33) for modal damping ratios](image)
CONCLUSIONS

Rubber bushings are commonly used for transverse stiffening of stay cables near the anchorages, and the influence of such stiffeners on the effectiveness of external dampers for stay-cable suppression has been investigated. Modeling the cable as a taut string, the bushing as a linear spring and the damper as a linear dashpot, an exact analytical solution for free vibrations has been obtained. An efficient iterative scheme has been developed for accurate determination of the complex eigenfrequencies, and an explicit asymptotic solution for the complex eigenfrequencies has been obtained, which is quite accurate in the first few modes for damper locations near the end of the cable. An “effective damper location”, which depends on the spring location and stiffness, emerged from the asymptotic solution, and using this effective damper location, the universal estimation curve for a linear viscous damper has been generalized to account for the influence of the bushing stiffness. The effect of the bushing stiffness is to reduce the effective distance of the damper from the end of the cable, reducing the maximum attainable damping ratio in each mode and increasing the corresponding optimal value of the external damper coefficient.

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