COMBINED EFFECTS OF FLEXURAL STIFFNESS AND AXIAL TENSION ON DAMPER EFFECTIVENESS IN SLENDER STRUCTURES

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ABSTRACT

Motivated by the problem of stay-cable vibration suppression in bridges, free vibrations of a tensioned beam with an intermediate viscous damper are investigated. Characteristic equations for the eigenvalues are derived for both pinned-pinned and clamped-clamped support conditions. For cases in which the eigenvalues are only slightly shifted by the damper, the eigenvalue loci have an asymptotic form similar to that previously observed for a taut cable with viscous damper. However, the maximum attainable modal damping ratios and the corresponding optimal damper coefficients are significantly affected by flexural stiffness, with the nature of this effect depending strongly on the type of support conditions.

Keywords: stay cable, viscous damper, vibration, flexural stiffness, complex eigenvalue problem

INTRODUCTION

The stays of many cable-stayed bridges have exhibited problematic large-amplitude vibrations, and to suppress these vibrations, supplemental dampers are commonly attached to the stays transversely near the anchorages. The dynamic behavior of the resulting cable-damper system has been the subject of much investigation in recent years, with the aim of facilitating effective damper design. Many of these studies have neglected the influence of flexural rigidity, often modeling the cable as a taut string (e.g., Pacheco et al. 1993), and in several cases including the effect of sag due to self-weight (e.g., Krenk and Nielsen 2002). Comparing the results of numerical calculation with a database of stay cable properties, Tabatabai and Mehrabi (2000) found that while the effect of sag was insignificant for most stay cables, the effect of flexural rigidity could be significant for many stays, particularly when the damper is located near the end of the cable, a condition that is often dictated by practical constraints. In this paper, the combined effects of flexural stiffness and axial tension on external damper effectiveness are investigated by considering the free vibrations of a tensioned beam with an intermediate viscous damper.

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PROBLEM FORMULATION

Free lateral vibrations are considered for a uniform axially loaded beam with viscous damper for two types of support conditions, as shown in Fig. 1. The beam is subjected to constant axial tension $T$ and has mass per unit length $m$ and flexural stiffness $EI$. The total span length of the beam is $\ell_0$, and the linear viscous damper with coefficient $c$ is attached at an intermediate point, dividing the beam into two segments of length $\ell_1$ and $\ell_2$. Because the support conditions are symmetric for both cases in Fig. 1, it is assumed without loss of generality that $\ell_1 \leq \ell_2$.

![Beam with damper](image)

**FIG. 1. Tensioned beam with viscous damper.**
(a) Pinned-pinned supports; (b) Clamped-clamped supports.

With no loading along the span, the equation of motion over each segment is given by the following well-known form, assuming small deflections in a single plane and neglecting shear deformation, rotary inertia, and internal damping:

$$EI \frac{\partial^4 y}{\partial x_j^4} - T \frac{\partial^2 y}{\partial x_j^2} + m \frac{\partial^2 y}{\partial t^2} = 0$$

(1)

where $y(x_j, t)$ is the transverse deflection and $x_j$ is the axial coordinate along the $j^{th}$ segment ($j = 1, 2$). The solution to (1) is represented in separable form as $y(x_j, t) = Y(x_j) e^{i\omega t}$, where $i = \sqrt{-1}$, and $\omega$ is complex in general, with its real part giving the frequency of the damped oscillation and its imaginary part giving the rate of decay. Substituting this form into (1) yields a fourth-order ordinary differential equation in the spatial coordinate for the eigenfunction $Y(x_j)$.

While the resulting solution for $Y(x_j)$ has often been expressed using a combination of hyperbolic and trigonometric functions, Main and Jones (2004a) note that improved numerical conditioning is achieved by expressing the solution in the following equivalent form:

$$Y(\bar{x}_j) = A_i \exp(-p\bar{x}_j) + A_2 \exp[-p(\ell_j - \bar{x}_j)] + A_3 \cos q\bar{x}_j + A_4 \sin q\bar{x}_j$$

(2)

where $\bar{x}_j = x_j / \ell_0$ is the nondimensional axial coordinate, $\ell_j = \ell_j / \ell_0$ is the nondimensional segment length, and $p$ and $q$ can be expressed as follows:
in which \( \hat{\omega} = \omega / [(\pi / \ell_0) \sqrt{T / m}] \) is a nondimensional frequency, normalized by the fundamental frequency of a taut string, and \( \xi = \sqrt{\pi \ell_0^2 / EI} \) is the square-root of the nondimensional axial tension. It follows from (3) that \( p^2 - q^2 = \xi^2 \) and that \( \hat{\omega} = pq / \pi \xi \). Tabatabai and Mehrabi (2000) report from a database of stay-cable properties that nearly all of the cables had values of \( \xi \) between 10 and 600, with 82% of the cables having \( \xi > 100 \).

**Pinned-Pinned Supports**

For the support conditions depicted in Fig. 1(a), the displacement is constrained to be zero at each pin support, while the slope at these locations may be nonzero. In contrast with the taut string (Krenk 2000, Main and Jones 2002), for which equilibrium of forces requires a discontinuity in slope at the damper location, the presence of nonzero flexural stiffness requires the slope \( dY / d\xi_j \) to be continuous across the damper. Because the damper applies no moment, the second derivative \( d^2Y / d\xi_j^2 \) is also continuous across the damper, while there is a jump in the third derivative at the damper location to balance the transverse damper force. Enforcing these requirements with the spatial form of solution in (2) leads eventually to the following characteristic equation (Main and Jones 2004a):

\[
0 = 1 \hat{\omega}^2 + \left[ p \cosh p \xi \right] \left[ Q_{pp}^j \frac{Q_{pp}^j}{Q_{pp}^1} + Q_{cp}^j \frac{Q_{cp}^j}{Q_{pp}^1} \right] = 0
\]

in which \( \hat{\xi} = c / \sqrt{Tm} \) is the nondimensional viscous damping coefficient, and the factors \( Q_{pp}^j \) and \( Q_{cp}^j \) are defined as follows:

\[
Q_{pp}^j = (1 - \ell_j^2) \sin q \ell_j
\]

\[
Q_{cp}^j = (1 + \ell_j^2) \sin q \ell_j - \frac{q}{p} (1 - \ell_j^2) \cos q \ell_j
\]

where \( \ell_j = \exp(-p \ell_j) \). The superscript PP in (5) denotes pinned-pinned supports, and the roots of the quantity \( Q_{pp}^j \) are the eigenvalues of the \( j^{th} \) segment with pinned-pinned supports. Similarly, the superscript CP in (6) denotes clamped-pinned supports, and the roots of \( Q_{cp}^j \) are the eigenvalues of the \( j^{th} \) segment with clamped-pinned supports. The subscript \( j = 0 \) in (4) denotes the total span, with nondimensional length \( \ell_0 = 1 \).

When \( \hat{\xi} = 0 \), (4) reduces to \( Q_{pp}^0 = 0 \), yielding the eigenvalues of the undamped pinned-pinned beam. The bracketed factor in the second term of (4) is the characteristic equation for the pinned-pinned beam with an intermediate pin support, discussed in detail in Main and Jones (2004b). Thus as \( c \to \infty \), the damper acts as an intermediate pin support,
constraining the displacement at the attachment point to zero, but providing no dissipation. The following equation can be obtained by rearranging (4) to isolate \( \hat{c} \), and then taking the real part of the resulting equation to eliminate \( \hat{c} \):

\[
\text{Re} \left[ \frac{(p^2 + q^2)Q_{0,pp}^{pp}}{p(Q_1^{pp}Q_2^{cp} + Q_1^{cp}Q_2^{pp})} \right] = 0
\]  

(7)

For prescribed values of \( \ell_1 \) and \( \xi \), (7) defines the loci of all possible values of the eigenvalues \( p \) and \( q \) in the complex plane, and it is analogous to the “phase equation” obtained by Main and Jones (2002) for the taut string with viscous damper.

**Clamped-Clamped Supports**

For the support conditions of Fig. 1(b), both the slope and the displacement are constrained to be zero at each support. The continuity conditions and the force balance at the damper location are the same as in the pinned-pinned case, and enforcing these conditions with the spatial form of solution in (2) yields the following characteristic equation (Main and Jones 2004a):

\[
Q_{0}^{CC} + \frac{i \xi \hat{c} p}{2(p^2 + q^2)}[Q_1^{cp}Q_2^{cc} + Q_1^{cc}Q_2^{cp}]
\]

(8)

in which \( Q_j^{cp} \) is given by (6), and the term \( Q_j^{cc} = Q_j^{cc,S}Q_j^{cc,A} \) is a product of two factors, which are given by

\[
Q_j^{cc,S} = (1 - \varepsilon_j) \cos \left( \frac{1}{2} q \ell_j \right) + \frac{q}{p} (1 + \varepsilon_j) \sin \left( \frac{1}{2} q \ell_j \right)
\]

(9)

\[
Q_j^{cc,A} = (1 + \varepsilon_j) \sin \left( \frac{1}{2} q \ell_j \right) - \frac{q}{p} (1 - \varepsilon_j) \cos \left( \frac{1}{2} q \ell_j \right)
\]

(10)

The superscript \( CC \) denotes clamped-clamped supports, and the roots of \( Q_j^{cc,S} \) (9) are the eigenvalues corresponding to the symmetric modes, while the roots of \( Q_j^{cc,A} \) (10) are the eigenvalues corresponding to the antisymmetric modes. Eq. (8) is analogous to (4) for the case of pinned-pinned supports. When \( \hat{c} = 0 \), (4) reduces to \( Q_0^{cc} = 0 \), yielding the eigenvalues of the undamped clamped-clamped beam. The bracketed factor in the second term of (8) is the characteristic equation for the clamped-clamped beam with an intermediate pin support, as discussed in Main and Jones (2004b). Thus as \( c \to \infty \), the damper acts as an intermediate pin support. The following equation can be obtained by rearranging (8) to isolate \( \hat{c} \) and then taking the real part of the resulting equation to eliminate \( \hat{c} \):

\[
\text{Re} \left[ \frac{(p^2 + q^2)Q_{0,pp}^{cc}}{p(Q_1^{cp}Q_2^{cc} + Q_1^{cc}Q_2^{cp})} \right] = 0
\]  

(11)

This equation is analogous to (7), defining the loci of all possible values of the eigenvalues \( p \) and \( q \) in the complex plane for prescribed values of \( \ell_1 \) and \( \xi \).
ASYMPTOTIC SOLUTION CHARACTERISTICS

An asymptotic form for the eigenvalue loci emerges in cases for which the frequency shift induced by locking the damper \( (c \to \infty) \) remains relatively small. In such cases, each eigenvalue locus traces a semi-circle in the complex plane that originates at the undamped frequency \( \omega_n^0 \) associated with \( c = 0 \) and terminates at the real-valued “clamped” eigenfrequency \( \omega_n^\infty \) associated with \( c \to \infty \). Similar asymptotic features were previously observed for a taut cable (Main and Jones 2002) and for a sagged cable (Krenk and Nielsen 2002) with a viscous damper. In such cases it is convenient to adopt the notation of Krenk and Nielsen (2002) to denote the complex-valued frequency increment induced by the damper in the \( n \)th mode:

\[
\Delta \omega_n = \omega_n - \omega_n^0; \quad \Delta \omega_n^\infty = \omega_n^\infty - \omega_n^0
\]  

(12)

where \( \Delta \omega_n^\infty \) denotes the real-valued frequency increment in the limit as \( c \to \infty \). Using this notation, the asymptotic semi-circular form of the eigenvalue loci can be expressed as follows:

\[
\frac{\text{Im}[\Delta \omega_n]}{\Delta \omega_n^\infty} \approx \sqrt{\Theta_n (1 - \Theta_n)}; \quad \text{where} \quad \Theta_n = \frac{\text{Re}[\Delta \omega_n]}{\Delta \omega_n^\infty}
\]  

(13)

The parameter \( \Theta_n \), referred to as the “clamping ratio” by Main and Jones (2002), is zero when \( c = 0 \) and approaches unity as \( c \to \infty \). Fig. 2 shows the asymptotic approximation (13) plotted along with exact values of the normalized frequency increment \( \Delta \omega_n / \Delta \omega_n^\infty \) in the first three modes for both pinned-pinned and clamped-clamped supports, computed numerically from (7) and (11) with \( \overline{t}_i = 0.02 \) and \( \xi = 50 \), which represent realistic values for a damped stay cable. In both cases, the agreement is very good, and Main and Jones (2004a) show that this agreement remains remarkably good over a wide range of \( \xi \) even in the limit of a beam without tension \( (\xi \to 0) \), provided the damper-induced frequency shifts remain small.

**FIG. 2: Normalized eigenfrequency loci** (\( \overline{t}_i = 0.02, \ \xi = 50 \)).

(a) Pinned-pinned supports; (b) Clamped-clamped supports.

A nondimensional design curve has been previously obtained for the taut string with a linear viscous damper near one end, relating the damping ratios in the first few modes to the viscous
damper coefficient. This “universal estimation curve” was originally identified from numerical computation by Pacheco et al. (1993) and the following analytical expression for this curve was later derived by Krenk (2000):

\[
\frac{\zeta_n}{\ell_1} \approx \frac{\pi \hat{c}n \bar{\ell}_1}{1 + (\pi \hat{c}n \bar{\ell}_1)^2}
\]

(14)

where the damping ratio is defined as \( \zeta_n = \text{Im} \left[ \frac{\omega_n}{\omega_n} \right] \). Eq. (14) predicts a maximum attainable damping ratio in mode \( n \) of \( \zeta_{n,\text{opt}} = \frac{\ell_1}{2} \), associated with a corresponding optimal nondimensional damper coefficient of \( c_{n,\text{opt}} = (\pi n \bar{\ell}_1)^{-1} \). This curve is of great utility in the design of viscous dampers for cable vibration suppression, and the influence of flexural stiffness on this relation and the associated optimal values is of significant practical interest.

Fig. 3 shows plots of \( \frac{\zeta_n}{\ell_1} \) against \( \hat{c}n \bar{\ell}_1 \) in the first mode for both pinned-pinned and clamped-clamped supports, with \( \bar{\ell}_1 = 0.02 \) and for three different values of \( \xi \). The curves corresponding to \( \xi = 1000 \) in Figs. 3(a) and 3(b) are virtually indistinguishable, and are in good agreement with (14). However, the evolution of these curves with increasing flexural stiffness (decreasing \( \xi \)) depends strongly on the nature of the support conditions. Decreasing \( \xi \) leads to significant increases in the maximum attainable damping ratio in the case of pinned-pinned supports, but leads to small decreases in the maximum attainable damping ratio in the case of clamped-clamped supports. For both types of support conditions, decreasing \( \xi \) leads to increases in the optimal damper coefficient, but this increase is much stronger in the case of clamped-clamped supports.

![Fig. 3](image-url)

**FIG. 3.** Influence of flexural stiffness on “universal estimation curve” \(( \bar{\ell}_1 = 0.02, \ n = 1 \). (a) Pinned-pinned supports; (b) Clamped-clamped supports.

Fig. 4 shows numerically computed values of \( \hat{c}_{n,\text{opt}}^n n \bar{\ell}_1 \) in the first mode plotted against the product \( \xi \bar{\ell}_1 \) for both pinned-pinned and clamped-clamped supports and for several different
values of $\bar{\ell}_i$. Only values of $\xi > 10$ are included in the plot, where $\xi = 10$ corresponds to the lower limit of for realistic stay cables, as noted previously. It is found that the results corresponding to different values of $\bar{\ell}_i$ collapse quite well along a single curve. While only values corresponding to the first mode are plotted in Fig. 5, the mode number $n$ is retained in the ordinate because values corresponding to the first few modes are observed to fall on the same curves as those for the first mode. This implies that curves corresponding to different modes shift uniformly along the abscissa with decreasing $\xi$, when plotted as in Fig. 3. Fig. 4 shows that the optimal damper coefficient is much more strongly affected by decreasing $\xi$ in the case of clamped-clamped supports than in the case of pinned-pinned supports. For both types of support conditions, the optimal damper coefficient is fairly close to the taut-string value for $\xi \bar{\ell}_i > 10$.

![Graphs showing influence of flexural stiffness on optimal damper coefficient](image)

**FIG. 4. Influence of flexural stiffness on optimal damper coefficient ($n = 1$, $\xi > 10$).**
(a) Pinned-pinned supports; (b) Clamped-clamped supports.

Fig. 5(a) shows numerically computed values of the normalized maximum attainable damping ratio $\frac{\zeta_{\text{opt}}}{\bar{\ell}_i}$ in the first mode plotted against the product $\xi \bar{\ell}_i$ for the case of pinned-pinned supports and for several different values of $\bar{\ell}_i$. As in Fig. 4, only values of $\xi > 10$ are included in the plot, and the values corresponding to different damper locations are well consolidated along a single curve. Fig. 5(a) shows that in the case of pinned-pinned supports the influence of bending stiffness can significantly increase the maximum attainable damping ratios above the taut-string values.

The situation is different in the case of clamped-clamped supports, and in Fig. 5(b) the numerically computed values of $\frac{\zeta_{\text{opt}}}{\bar{\ell}_i}$ in the first mode are plotted directly against $\xi$, rather than against the product $\xi \bar{\ell}_i$ as in Fig. 5(a). In the case of clamped-clamped supports, the ratio $\frac{\zeta_{\text{opt}}}{\bar{\ell}_i}$ is seen to be relatively constant with $\xi$, and a linear scale is used on the ordinate rather than a logarithmic scale as in Fig. 5(a). In the range between 10 and 1000, encompassing nearly all stay cables, the influence of bending stiffness is seen to produce slight reductions of the maximum damping ratio below the taut-string values. Further reductions of $\xi$ lead to increases
in the maximum attainable damping ratio, tending to a value of about 0.8 as $\xi \to 0$. Fig. 5 clearly shows that in this asymptotic regime, more damping can be added to a given mode with pinned-pinned supports than with clamped-clamped supports.

**FIG. 5. Influence of flexural stiffness on optimal first-mode damping ratios.**
(a) Pinned-pinned supports; (b) Clamped-clamped supports.

**CONCLUSIONS**

The combined effects of flexural stiffness and axial tension in a beam on the effectiveness of an external viscous damper have been investigated. For cases in which the frequency shift induced by locking of the damper remains small, the eigenfrequencies were observed to trace semi-circular loci in the complex plane, with this simple form persisting over a wide range of nondimensional axial tension. However, flexural stiffness was observed to significantly influence the maximum attainable damping ratios and the corresponding optimal damper coefficients, with the nature of this effect depending strongly on the type of support conditions.

**REFERENCES**


